Mimetic Discretizations of Diffusion Equation on Polygonal Meshes

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As mathematical modeling of fluid flow becomes more sophisticated, the need for discretization methods handling meshes with mixed types of elements is arisen.

Practice experiences show that the most effective discretization methods preserve and mimic the underlying properties of original continuum differential operators. Conservation laws, solution symmetries, and the fundamental identities and theorems of vector and tensor calculus are example of such properties. We developed a new family of mimetic discretizations for diffusion-type equations (e.g. pressure equation in porous medium applications) on general polygonal meshes:

$$\begin{aligned} \text{div } \mathbf{u} &= Q, \\ \mathbf{u} &= -\mathbf{K} \text{ grad } p. \end{aligned}$$

Here p and $\mathbf{u}$ denote the fluid pressure and velocity, respectively, $\mathbf{K}$ denotes a full tensor representing the rock permeability divided by the fluid viscosity, and Q denotes a source function.

The discretizations we developed are locally conservative and use discrete flux, $\mathcal{G}$, and divergence, $\mathcal{DIV}$, operators (for continuum operators $-\mathbf{K}$grad and div, respectively) which are adjoint to each other, i.e. $\mathcal{G} = \mathcal{DIV}^*$.

The class of polygonal meshes is very wide and includes meshes used in many applications. For instance, locally refined meshes with hanging nodes are among them. A hanging node occurs when two (or more) elements share an edge with one element. If we consider the hanging node as an additional vertex of that element, we get a polygonal mesh containing degenerate polygons, i.e. polygons with angles equal to 180° between some adjacent edges. Another example comes from applications with non-matching

meshes. The non-matching mesh occurs when two different meshes are used in two adjacent subdomains. If we consider all the mesh points on the interface between the subdomains as vertices of a conformal mesh, we get again a polygonal mesh with possibly non-convex polygons.

Nowadays, the use of polygonal meshes is limited by a small number of accurate discretization schemes. We mention here the finite volume scheme developed by T.Palmer [1]. The scheme is exact for uniform flows (constant flux and linear pressure) but results in a non-symmetric coefficient matrix. Therefore, it requires the use of non-traditional iterative solvers. On contrary, the new mimetic discretization results in an algebraic problem with a symmetric positive definite matrix. Therefore, the problem may be solved with the conjugate gradient method. We believe that the developed mimetic finite difference discretizations will make polygonal meshes more attractive for engineering applications in the future.
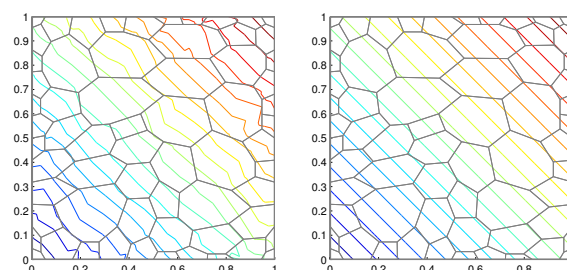
Isobars in the uniform flow for the old (left) and new (right) mimetic discretization methods.

The developed methodology is based on the divide and conquer principle. First, we consider each mesh polygon as an independent domain and generate an independent discretization for this polygon. Second, the system of element-based discretizations is closed by imposing boundary conditions and continuity conditions for the fluid pressure and normal velocity component on polygon edges.

We proved that the resulting mimetic discretization is exact for uniform flows. In the figure, we demonstrate the superiority of the new discretization over the old one [2] which does not preserve uniform flows. For the case of general

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flows, we demonstrated with numerical experiments [3] the second order convergence rate for the fluid pressure and the first order convergence rate for the fluid velocity. It confirms the convergence rates observed in other types of lower order discretizations on non-smooth meshes.

Precise calculation of the fluid velocity is important for porous medium and some other applications. The points or lines where the numerical solution is super-close to the exact solution may be used to improve the accuracy of the overall simulation. Depending on the mesh smoothness and solution regularity, the convergence rate for the fluid velocity varies between the first and the second order.

Acknowledgements

References

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